A GENERALIZATION OF WEIERSTRASS' PREPARATION THEOREM FOR A POWER SERIES IN SEVERAL VARIABLES*

BY

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A number of proofs of the preparation theorem for a single power series in several variables have recently been published.† The theorem has to do with a series $F(x_1, x_2, \cdots, x_m : y)$ for which $F(0, 0, \cdots, 0 : y)$ begins with a term of degree ν in y, and the statement is that there exists a unique multiplier $M(x_1, x_2, \cdots, x_m : y)$ such that the product MF is a polynomial of the form

$$MF = y^\nu + a_1 y^{\nu-1} + a_2 y^{\nu-2} + \cdots + a_{\nu-1} y + a_\nu . \tag{1}$$

The coefficients $a(x_1, x_2, \cdots, x_m)$ are power series in $x_1, x_2, \cdots, x_m$ which have no constant terms, and the multiplier M is a series with constant term different from zero. If the original series converges, then the M and the a's converge. It follows readily that the roots of $F = 0$ which lie in a suitably chosen domain

$$|x_i| < \epsilon, \quad |y| < \epsilon \qquad (i = 1, 2, \cdots, m), \tag{2}$$

are precisely the roots of the polynomial so determined, and if δ is sufficiently small and

$$|x_i| < \delta \qquad (i = 1, 2, \cdots, m),$$

then all of the roots of the polynomial lie in the region (2) and are roots of F.‡

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† Goursat, Bulletin de la Société Mathématique de France, vol. 36 (1908), p. 209; Bliss, Bulletin of the American Mathematical Society, 2d Series, vol. 16 (1910), p. 356; MacMillan, Bulletin of the American Mathematical Society, 2d Series, vol. 17 (1910), p. 116. These are algebraic in character. The older proofs were based on the theory of functions of a complex variable. See, for example, Weierstrass, *Werke*, vol. 2, p. 135; Poincaré, *Thèse*, p. 5 ff.; Picard, *Traité d'Analyse*, vol. 2, p. 241; Goursat, *Cours d'Analyse Mathématique*, vol. 2, p. 280.

‡ Professor W. F. Osgood recently called my attention to the fact that Cauchy was in possession of the principal properties of the solutions of the equation $F(x, y) = 0$ in the neighborhood of the origin as early as 1832 (see his Exercices d'Analyse Mathématique (1841), vol. 2, p. 52). Cauchy showed that the number of roots y corresponding to any value of x in a sufficiently small neighborhood of $x = 0$ is the same as the number ν which correspond to $x = 0$ itself. Furthermore he found that the sum

$$f(y_1) + f(y_2) + \cdots + f(y_\nu).$$

POINCARÉ inferred, with the help of the preparation theorem, that the values of $y_1, y_2, \cdots, y_n$ which belong to solutions in the neighborhood of the origin for a system of equations

$$(3) \qquad F_\alpha(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n) = 0 \qquad (\alpha = 1, 2, \cdots, n)$$

in which the first members are series without constant terms, are all algebroid functions of the x's, provided that the functions F_α are "distinct." In other words each y_α belonging to such a solution is a root of a polynomial of the type (1). His method, however, does not permit one to say that for every root of the polynomial so found there exists a corresponding root of the system of equations, and in fact his polynomial has in general extraneous roots which do not correspond to solutions. The situation is similar to the algebraic one which arises when the functions F_α are polynomials in $y_1, y_2, \cdots, y_n$. It is then comparatively easy to show that the element y_n of any solution of the equations $F_\alpha = 0$ must be a root of a polynomial in y_n whose coefficients are rational in those of the F_α, but it is a more complicated matter to show the existence of a polynomial of a sufficiently low degree having also the converse property that any root of it belongs to a solution of the system.

It is the purpose of this paper to show that a polynomial in y_n of the type (1) can in general be determined for equations (3), such that any solution of these equations lying in a suitably chosen region

$$(4) \qquad |x_i| < \epsilon, \quad |y_\alpha| < \epsilon \qquad (i = 1, 2, \cdots, m : \alpha = 1, 2, \cdots, n)$$

provides a root $(x_1, x_2, \cdots, x_m : y_n)$ of the polynomial, and for all values of the x's in a certain region

$$|x_i| < \delta \qquad (i = 1, 2, \cdots, m)$$

the roots of the polynomial satisfy the inequalities (4) and each of them belongs to a solution $(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n)$ of equations (3) which also lies in (4). If the degrees of the homogeneous polynomials $f_\alpha(y_1, y_2, \cdots, y_n)$ of lowest degree in the series $F_\alpha(0, 0, \cdots, 0 : y_1, y_2, \cdots, y_n)$ are ν_α, then the degree of the polynomial in y_n is the product $N = \nu_1 \nu_2 \cdots \nu_n$. To any set of values in a sufficiently small region about the origin, there correspond therefore exactly N solutions of equations (3), and the elements y_α of these solutions are algebroid functions of $x_1, x_2, \cdots, x_m$. The only con-

where f is any analytic function and $y_1, y_2, \cdots, y_\nu$ the roots of $F = 0$ corresponding to a given x, is a single valued analytic function of x. It follows readily that any symmetric function of the roots is expressible as a series in x vanishing when $x = 0$, since any such function can be expressed in terms of the sums

$$y_1^p + y_2^p + \cdots + y_\nu^p$$

for different values of p. The proof of the preparation theorem referred to above, which GOURSAT gives in his *Cours d'Analyse*, follows CAUCHY's method of procedure.

ditions on the coefficients of the F_α essential to the proof of the theorem are that the series F_α converge, and that the resultant R of the polynomials f_α shall be different from zero.

KISTLER * has considered the solutions of any set of equations

$$F_\alpha(x_1, x_2, \cdots, x_m) = 0 \qquad (\alpha = 1; 2, \cdots, n)$$

in which the functions F are analytic. By an extension of Kronecker's algorithm for the case when the functions F are polynomials, he shows that the solutions of such a system can be classified into "Mannigfaltigkeiten" of dimensions greater than or equal to $m - n$. Furthermore each "Mannigfaltigkeit" separates into a number of "analytische Gebilde" for which the variables $x_1, x_2, \cdots, x_m$ are expressible as analytic functions of k parameters u, where k is the dimension of the Mannigfaltigkeit. The theorem of the present paper has a relationship to these results which is similar to that of the elimination theory to Kronecker's algorithm in algebra.

In the first volume of his *Mécanique Céleste* (page 72) POINCARÉ makes the statement that the solutions of a system of equations (3), at least so far as the solutions in the neighborhood of the origin are concerned, is equivalent to the solution of a system

$$(5) \qquad \varphi_\alpha(x_1, x_2, \cdots, x_m : y_1, y_2, \cdots, y_n) = 0 \qquad (\alpha = 1, 2, \cdots, n),$$

in which the φ_α are polynomials in $y_1, y_2, \cdots, y_n$. The proof of this theorem has been considered recently by W. D. MACMILLAN. By applying the usual algebraic elimination theory to the polynomials φ_α, a polynomial $\Phi(x_1, x_2, \cdots, x_m : y_n)$ would be found for which the lowest term in $\Phi(0, 0, \cdots, 0 : y_n)$ would be, say, of degree ρ. By applying the preparation theorem to $\Phi(x_1, x_2, \cdots, x_m : y_n)$, therefore, a polynomial Ψ of type (1) of degree ρ would appear. This is not in general the polynomial whose existence is sought in this paper, as may readily be shown by means of an example. The polynomials φ_α may have roots for which y_n is near to zero, while the values of $y_1, y_2, \cdots, y_{n-1}$ are not small. Such values of y_n must appear as roots of the polynomial Ψ, and its degree ρ is therefore in general too great.†

* *Über Funktionen von mehreren komplexen Veränderlichen,* Dissertation, Göttingen (1905).

† The polynomials

$$y^2 + z^2 - 2y - 2z + x = 0, \quad 2y^2 - z^2 - 4y + 2z = 0$$

have only one solution for which y and z are power series in x vanishing with x. The resultant of these two equations after the elimination of y is, however,

$$9z^4 - 36z^3 + 12(x + 3)z^2 - 24xz + 4x^2.$$

If the preparation theorem is applied to this, a polynomial in z of degree 2 is found, one of the roots of which must therefore correspond to a solution (y, z) for which y is not in the neighborhood of the origin. The same is true of the polynomial in y found by eliminating z.

The problem of proving the theorem of §1 for a system of polynomials (5) seems to involve precisely the difficulties inherent to the more general case of the system (3).

In concluding these introductory remarks it is interesting to note that the theorem which is to be proved is not only a generalization of the preparation theorem for a single power series, but is also an extension of the well-known theorem concerning implicit functions. If the " characteristic " polynomials * f are all of degree one, then their determinant is the resultant R, and the polynomials (1) for the various elements y_α are all of the first degree.

The theorem determines the number of solutions of equations (3) in the neighborhood of the origin to be N, provided that the resultant R is different from zero, and affords a basis for definition of the multiplicity of roots and the discriminant of the system of equations $F_\alpha = 0$ in the neighborhood of the origin.

§ 1. *The existence theorem stated with some of its consequences.*

In the following pages the notations x, y, F, and so on, will be used as row letters to indicate the sets of symbols

$$x = (x_1, x_2, \cdots, x_m), \qquad y = (y_1, y_2, \cdots, y_n),$$
$$F = (F_1, F_2, \cdots, F_n).$$

The number of elements in the various sets will be clear from the context in any particular case.

Let the series $F(x:y)$ be formal series, that is, series with literal coefficients, having for $x = 0$ characteristic polynomials $f_\alpha(y)$ of degree ν_α, respectively. The theorem to be proved is the following:

There exists a polynomial

$$P(x:y_n) = y_n^N + a_1 y_n^{N-1} + \cdots + a_{N-1} y_n + a_N$$

of degree $N = \nu_1 \nu_2 \cdots \nu_n$ *with the following properties. The functions* $a_p(x_1, x_2, \cdots, x_m)$ *are series vanishing with* x *and having coefficients which are rational in the coefficients of the original series* F. *The only denominators which occur in these rational expressions are powers of the resultant* R *of the characteristic polynomials* f. *The polynomial* $P(x:y_n)$ *has the following properties in any numerical case, that is, when the literal coefficients of the function* F *are replaced by numbers:* (1) *if the coefficients are so chosen that the series* F *are convergent and* $R \neq 0$, *then* $P(x:y_n)$ *is convergent;* (2) *if* $(\xi:\eta)$ *is a root of the system* $F = 0$ *lying in a sufficiently small region*

$$(6) \qquad |x| < \epsilon, \quad |y| < \epsilon$$

then $(\xi:\eta_n)$ *is a root of* $P(x:y_n) = 0$; (3) *there exists a positive constant*

* This nomenclature is due to MacMillan.

$\delta < \epsilon$ such that to any set of values ξ satisfying the inequalities

$$|x| < \delta \tag{7}$$

and any root $(\xi : \eta_n)$ of $P(x : y_n) = 0$ there correspond values $(\eta_1, \eta_2, \cdots, \eta_{n-1})$ which with $(\xi : \eta_n)$ satisfy the inequalities (6) *and form a solution $(\xi : \eta)$ of the system $F = 0$.*

The proof which is to be made in the succeeding section is an inductive one. It is legitimate therefore to suppose the theorem true for n functions F containing n dependent variables y and any number of independent variables x, and to study some of its consequences in that case. In the first place

The function $P(x : y_n)$ is formally irreducible, that is, it cannot be the product of two factors of type similar to its own.

For if it were the product of two such factors it would be reducible also for any special numerical choice of the coefficients of the functions F. Let F_a be a homogeneous polynomial of degree ν_a in $y_1, y_2, \cdots, y_n$ and a single independent variable x_1, with coefficients so chosen that $R \neq 0$ and the resultant polynomial of $F_1, F_2, \cdots, F_n$ is irreducible. The corresponding polynomial $P(x : y_n)$ would then be, apart from a constant factor, identically equal to that resultant, since the roots of the two are the same for all values of x_1, and hence $P(x : y_n)$ could not be formally the product of two factors. If it were, each factor would have to be a polynomial in x_1, y_n, and this would contradict the possibility of choosing the coefficients of the F's in this special case so that their resultant is irreducible.

The polynomial $P(x : y_n)$ is the only one, with coefficients rational in those of the functions F and having only a finite number of different denominators, which has the properties described above.

Suppose that there were another $\bar{P}(x : y_n)$. The discriminant D of $P(x : y_n)$ is a series in x with coefficients like those of P itself. It does not vanish formally identically since P is formally irreducible. Furthermore P and $\bar{P}$ have at least one pair of corresponding coefficients formally distinct. It is possible then to choose numerical values for the coefficients of the functions F so that D is not identically zero in x, so that all the coefficients in P and $\bar{P}$ are finite, and so that at least one coefficient in P is different from the corresponding one in $\bar{P}$. It follows from this that $x = \xi$ can be so selected numerically that the discriminant D is different from zero, and at least one function a_p in P is different from the corresponding function a_p in $\bar{P}$. The roots of P and $\bar{P}$ could not then all be the same, as they should be by the properties (2) and (3).

The polynomial $P(x : y_n)$ is linearly expressible in the form

$$P(x : y_n) \equiv M_1F_1 + M_2F_2 + \cdots M_nF_n, \tag{8}$$

where the multipliers M are series in x, y with coefficients rational in the coefficients of the functions F like those of P itself.

Consider the functions $F_\alpha - v_\alpha$ $(\alpha = 1, 2, \cdots, n)$ in the variables v, x, y. According to the theorem a polynomial $P(v, x : y_n)$ exists with properties analogous to those of $P(x : y_n)$, and $P(0, x : y_n)$ must be $P(x : y_n)$ itself since the latter is unique.

The series of power series $P(F, x : y_n)$ may be developed formally in powers of x, y, and it can readily be seen that its coefficients have the properties of those of $P(x : y_n)$. It must, however, be formally identically zero. If not, a set of numerical coefficients could be selected for the functions F for which $P(F, x : y_n)$ would not be identically zero. It is convergent by Weierstrass' theorem concerning the sum of an infinite number of power series, when the functions F and $P(v, x : y_n)$ are convergent. Choose now any set $(x : y) = (\xi : \eta)$ whatsoever for which $(\xi : \eta)$ lies in the domain of convergence of the F_α and $P(F, x : y_n)$, and so small that the values $\varphi_\alpha = F_\alpha(\xi : \eta)$ with ξ and η lie in the domain of convergence of $P(v, x : y_n)$. Then

$$P[F(\xi : \eta), \quad \xi : \eta_n] = P(\varphi, \xi : \eta_n) = 0,$$

since $P(v, x : y_n)$ has the properties described in the theorem with respect to the series $F_\alpha - v_\alpha$, and the φ's have been so determined that η is a solution of the equations $F(\xi ; y) - \varphi = 0$. Hence the assumption that $P(F, x : y_n)$ is not identically zero is contradicted.

The series $P(v, x : y_n)$ can be written in the form

$$P(v, x : y_n) = P(0, x : y_n) + m_1 v_1 + m_2 v_2 + \cdots + m_n v_n,$$

where the m_α are series in v, x, y. If the substitution $v = F$ is made, it follows readily from what precedes that $P(x : y_n) = P(0, x : y_n)$ is expressible in the form (8).

Consider now the set of functions $G(t, x : z)$ formed from the functions F by substituting

$$y_1 = z_1, \quad y_2 = z_2, \quad \cdots, \quad y_{n-1} = z_{n-1},$$

$$y_n = -t_1 z_1 - t_2 z_2 - \cdots - t_{n-1} z_{n-1} + z_n.$$

Since the theorem is supposed to hold for any number of independent variables, there will be a corresponding polynomial $Q(t, x : z_n)$ of degree N in z_n, which has properties with respect to the series G analogous to those described in the theorem and reduces to $P(x : y_n)$ for $t = 0$, since the characteristic polynomials for the functions G found by putting $t = z = 0$ are exactly the characteristic polynomials $f(z)$ of the functions F.

For any numerical choice of the coefficients in F for which these functions converge, the function $Q(t, x : z_n)$ will also be convergent. There will exist two

positive constants, ϵ *and* $\delta \leqq \epsilon$, *such that to any root* $(\xi : \eta)$ *of the system* $F = 0$ *in the region*

$$(9) \qquad | x | < \epsilon, \quad | y | < \epsilon$$

there will correspond a factor of $Q(t, x : z_n)$ *of the form*

$$(10) \qquad z_n - t_1\eta_1 - t_2\eta_2 - \cdots - t_{n-1}\eta_{n-1} - \eta_n,$$

and for any value of ξ *which satisfies the inequalities*

$$| x | < \delta$$

the polynomial $Q(t, x : z_n)$ *will be completely factorable into linear factors of the form just written, for each of which the set* $(\xi : \eta)$ *lies in the region* (9) *and is a root of the equations* $F = 0$.

For any numerical choice of the coefficients of F which make these functions converge, the series $G(t, x : y)$ and consequently $Q(t, x : z_n)$ will also be convergent. If the domain for G and Q analogous to (6) is

$$(11) \qquad | t | < \epsilon, \quad | x | < \epsilon, \quad | z | < \epsilon,$$

then for any root $(\xi : \eta)$ of the functions F satisfying these inequalities the values

$$z_1 = \eta_1, \quad z_2 = \eta_2, \quad \cdots, \quad z_{n-1} = \eta_{n-1}, \quad z_n = t_1\eta_1 + t_2\eta_2 + \cdots t_{n-1}\eta_{n-1} + \eta_n$$

will satisfy the equations $G = 0$, and the expression (10) will necessarily be a factor of the function $Q(t, x : z_n)$.

On the other hand suppose that $x = \xi$ satisfies the inequalities

$$(12) \qquad | t | < \delta, \qquad | x | < \delta$$

analogous to (7) for the functions G and Q. If the discriminant of $Q(t, \xi : z_n)$ vanishes identically in t, the highest common factor of Q and $\partial Q / \partial z_n$ can be divided out of Q leaving a polynomial $\bar{Q}$ of the same type.* Choose now $t = \tau$ in the region (12) so that the discriminant of $\bar{Q}$ is different from zero. According to the property (3) there will correspond to each root ζ_n of $\bar{Q}(\tau, \xi; z_n)$ a root system $(\tau, \xi : \zeta)$ of the equations $G = 0$. The value of δ in the inequalities (12) can be taken so small that the n quantities

$$\eta_1 = \zeta_1, \ \eta_2 = \zeta_2, \ \cdots, \ \eta_{n-1} = \zeta_{n-1}, \ \eta_n = -\tau_1\zeta_1 - \tau_2\zeta_2 - \cdots - \tau_{n-1}\zeta_{n-1} - \zeta_n$$

lie in the domain of convergence of the functions F. From the identities

$$F(x : y) = G(t, x : z)$$

* The coefficients of the highest common factor are rational in t. If the coefficient of the highest power of z_n is made to be unity by dividing by a suitable function of t, then the other coefficients are rational and integral in t. Otherwise some of the roots of Q would become infinite at particular values of t in its domain of convergence.

it follows that $(\xi : \eta)$ satisfies the equations $F = 0$. From the same identity it follows that the values

$$z_1 = \eta_1,\ z_2 = \eta_2,\ \cdots,\ z_{n-1} = \eta_{n-1},\ z_n = t_1\eta_1 + t_2\eta_2 + \cdots + t_{n-1}\eta_{n-1} + \eta_n$$

in which the values of t are undetermined, make the functions $G(t, \xi : z)$ vanish identically, and $\bar{Q}(t, \xi : z_n)$ must therefore have the factor (10). The roots of $\bar{Q}(\tau, \xi : z_n)$ are all distinct, and hence the factors of the form (10) corresponding to the different roots are all distinct. It follows that the factors of $Q(t, \xi : z_n)$ are all of this form.

The function $Q(t, x : z_n)$ is formally a polynomial of order N in the variables $t_1, t_2, \cdots, t_{n-1}, z_n$.

Suppose that Q contained a term of higher degree than N in t, z_n. The coefficients of the functions F could be chosen numerically so that this coefficient would remain numerically different from zero, and there would exist a ξ satisfying the stipulations of the proof just preceding. But for such a value ξ it has just been shown that $Q(t, \xi : z_n)$ is a polynomial of degree N.

The coefficients η in the factors of $Q(t, \xi : z_n)$ are algebroid functions of $\xi_1, \xi_2, \cdots, \xi_m$ of order N.

For, according to what precedes, the set $(\xi : \eta)$ associated with any factor of $Q(t, \xi : z_n)$ is a root of the equations $F = 0$. The theorem at the beginning of this section implies the existence of a polynomial $P_\alpha(x : y_\alpha)$ for y_α similar to $P(x : y_n)$, and for each element η_α of a root of the equations $F = 0$ the corresponding polynomial $P_\alpha(\xi : \eta_\alpha)$ must vanish. Consequently η_α is an algebroid function of the elements ξ.

Suppose now that the N systems of variables $(y_{1p}, y_{2p}, \cdots, y_{np})$ $(p = 1, 2, \cdots, N)$ are so related to the variables x that

$$Q(t, x : z_n) = \prod_{p=1}^{N} (z_n - t_1 y_{1p} - t_2 y_{2p} - \cdots - t_{n-1} y_{n-1\,p} - y_{np}).$$

Any one of the so-called elementary symmetric functions * of degree ρ of the N systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ must then be equal to one of the series in x which occurs as a coefficient in $Q(t, x : z_n)$ regarded as a polynomial in t, z_n, since each of these symmetric functions occurs as a coefficient in the polynomial on the right. It is in fact one which occurs in the coefficient of $z_n^{N-\rho}$. Any rational integral homogeneous symmetric function of the root systems of degree ρ is equal to a rational integral expression in the elementary symmetric functions, and the sum of the degrees of the elementary functions in each term is ρ. Hence it follows that

* An elementary symmetric function of the N systems is one which is rational, integral, and linear in the elements of each set. For the definition of an elementary symmetric function and the theorems used in the text concerning the representation of other symmetric functions in terms of them, see König, *Einleitung in die allgemeine Theorie der algebraischen Grössen*, pp. 312 ff.

Any rational integral symmetric function of the N root systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ $(p = 1, 2, \cdots, N)$ *corresponding to the factors of* $Q(t, x : z_n)$, *is expressible as a series in* x *without a constant term.*

It is important for later deductions to consider the case in which the functions F and G contain only one independent variable $x_1 = y_{n+1}$ which enters with $y_1, y_2, \cdots, y_n$ in such a way that the leading polynomials $f_\alpha(y_1, y_2, \cdots, y_{n+1})$ with respect to all the variables are homogeneous and of degree ν_α. Let y be replaced by uy in the functions G and Q, and denote the system of functions $G_\alpha / u^{\nu_\alpha}$ by $\bar{G}$. The polynomial $\bar{Q}$ corresponding to the $\bar{G}$ is exactly Q / u^N. For this latter polynomial satisfies the conditions (2) and (3) for the functions $\bar{G}$ in numerical cases for which $u \neq 0$, and by an argument similar to that used in proving the uniqueness of the polynomial P it follows that $\bar{Q}$ and Q / u^N must be equal identically. As u approaches zero, $\bar{Q}$ and consequently Q / u^N remains finite and reduces finally for $u = 0$ to the corresponding resultant polynomial for the functions $f(y_1, y_2, \cdots, y_{n+1})$. In other words the terms of $Q(t, y_{n+1} ; z_n)$ of lowest degree in z_n, y_{n+1} are homogeneous and of degree N in these variables. The terms of lowest degree in y_{n+1} in the expression for one of the rational integral homogeneous symmetric functions of degree ρ mentioned above, will in this case be exactly the term of degree ρ arising out of the process of elimination for the polynomials f.

2. *The proof of the theorem.*

The preparation theorem stated in the introduction is precisely the theorem described in the preceding section for a single function. Suppose then that the latter is true for n functions in n dependent variables y and any number of independent variables x. It is to be proved for $n + 1$ functions

$$F_\beta(x : y_1, y_2, \cdots, y_n, y_{n+1}) \quad (\beta = 1, 2, \cdots, n+1)$$

for which the characteristic polynomials are $f_\beta(y_1, y_2, \cdots, y_n, y_{n+1})$ of degrees ν_β respectively.

Consider first the series $G(t, x, y_{n+1} : z)$ and $Q(t, x, y_{n+1} : z_n)$ for the functions F_α $(\alpha = 1, 2, \cdots, n)$, in which x, y_{n+1} are to be thought of as the independent variables. The product series

$$(13) \qquad \prod_{p=1}^{N} F_{n+1}(x : y_{1p}, y_{2p}, \cdots, y_{np}, y_{n+1}) = H(x : y_{n+1})$$

is symmetric in the $N = \nu_1 \nu_2 \cdots \nu_n$ systems $(y_{1p}, y_{2p}, \cdots, y_{np})$, and with the results of the preceding section in mind it is seen that it can be expressed, formally at least, as a series in x, y_{n+1} with coefficients rational in those of $F_1, F_2, \cdots, F_{n+1}$, the only denominators being powers of R, where R is the resultant of the characteristic polynomials $f_\alpha(y_1, y_2, \cdots, y_n, 0)$ $(\alpha = 1,$

$2, \cdots, n$) of the first n functions, with respect to $y_1, y_2, \cdots, y_n$. When $x = 0$ the terms of the lowest degree in $(y_{1p}, y_{2p}, \cdots, y_{np}, y_{n+1})$ $(p = 1, 2, \cdots, N)$ are equal to the product

$$\prod_{p=1}^{N} f_{n+1}(y_{1p}, y_{2p}, \cdots, y_{np}, y_{n+1})$$

of degree $N_{n+1} = N\nu_{n+1}$. When the symmetric functions of the systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ are replaced by the corresponding series in y_{n+1}, this expression has for its term of lowest degree in y_{n+1} exactly

$$\frac{R_{n+1}}{R^{\nu_{n+1}}} y_{n+1}^{N_{n+1}},$$

where R_{n+1} is the resultant of the polynomials $f_\beta(y_1, y_2, \cdots, y_n, y_{n+1})$ $(\beta = 1, 2, \cdots, n+1)$,* since the series representing symmetric functions of $(y_{1p}, y_{2p}, \cdots, y_{np})$ of degree ρ begin each with a term of degree ρ in y_{n+1}, which is precisely the same as if the series $F_1, F_2, \cdots, F_n$ were the homogeneous polynomials $f_\alpha(y_1, y_2, \cdots, y_n, y_{n+1})$ $(\alpha = 1, 2, \cdots, n)$. Hence the preparation theorem may be applied to $H(x : y_{n+1})$, and the resulting polynomial $P_{n+1}(x : y_{n+1})$ of degree N_{n+1} in y_{n+1} will be the one desired in the theorem.

It is necessary next to show that $P_{n+1}(x : y_{n+1})$ has the properties described in the preceding section. In the first place its coefficients are rational in those of the functions F_β with denominators involving at most powers of R and R_{n+1}, and it will appear later that R_{n+1} is the only one of these functions which occurs.

That the series will be convergent for any numerical choice of the coefficients of F_β for which these functions converge and for which the denominators in the coefficients of P_{n+1} are different from zero, can be shown as follows. In such a case two positive constants δ and ϵ $(\delta \leqq \epsilon)$ exist such that in the regions

$$|x| < \epsilon, \quad |y| < \epsilon, \tag{14}$$

$$|x| < \delta, \quad |y_{n+1}| < \delta, \tag{15}$$

the solutions of the equations $F_\alpha = 0$ $(\alpha = 1, 2, \cdots, n)$ have the properties described in §1. The regions can be taken so small that for any $(\xi : \eta_{n+1})$ in (14) the corresponding root systems $(\eta_{1p}, \eta_{2p}, \cdots, \eta_{np})$ of $F_1, F_2, \cdots, F_n$ are all in the convergence region of F_{n+1}, and furthermore so that for $|\xi| < \delta$ the series in ξ which replace the symmetric functions of $(\eta_{1p}, \eta_{2p}, \cdots, \eta_{np})$ are also convergent. The theorem of Weierstrass concerning the uniformly convergent sum of an infinite number of power series can therefore be applied

* See KÖNIG, loc. cit., p. 311.

to show that $H(x:y_{n+1})$ is convergent. It follows that $P_{n+1}(x:y_{n+1})$ also converges.

A new constant $\epsilon_{n+1} \leqq \delta$ can now be chosen so small that any zero of $H(x:y_{n+1})$ satisfying the inequalities

$$(16) \qquad |x| < \epsilon_{n+1}, \quad |y| < \epsilon_{n+1},$$

must make $P_{n+1}(x:y_{n+1})$ vanish also, and vice versa. Any root $(\xi;\eta)$ of the system $F_\beta = 0$ $(\beta = 1, 2, \cdots, n+1)$ in the region (16) must be one of the root systems of the functions F_α $(\alpha = 1, 2, \cdots, n)$ for the corresponding values $(\xi;\eta_{n+1})$, and hence must make $H(x:y_{n+1})$ and therefore $P_{n+1}(x:y_{n+1})$ vanish. Furthermore if δ_{n+1} is sufficiently small the roots of $P_{n+1}(x:y_{n+1})$ corresponding to a value ξ satisfying

$$|x| < \delta_{n+1}$$

will all lie in the region (16). If η_{n+1} is such a root, then one at least of the root systems $(\eta_{1p}, \eta_{2p}, \cdots, \eta_{np})$ corresponding to $(\xi:\eta_{n+1})$ for the equations $F_\alpha = 0$ must make F_{n+1} vanish, since $P_{n+1}(x:y_{n+1})$ and $H(x:y_{n+1})$ are both zero.

From the method employed in deriving the polynomial $P_{n+1}(x:y_{n+1})$ it would seem that the coefficients involve the resultant R of the functions $f_\alpha(y_1, y_2, \cdots, y_n, 0)$ $(\alpha = 1, 2, \cdots, n)$ in at least some of the denominators. This is, however, not the case. If it were, there would be a coefficient of the form $A / R^a R^b_{n+1}$, where A is a polynomial in a finite number of the coefficients of the F_β's, and a and b are positive integers. If instead of $(F_1, F_2, \cdots, F_n)$ another set, say $(F_1, F_2, \cdots, F_{n-1}, F_{n+1})$ had been used in deriving P_{n+1}, the corresponding coefficient would have had the form $B / S^c R^d_{n+1}$ where S is the resultant of $f_\gamma(y_1, y_2, \cdots, y_n, 0)$ $(\gamma = 1, 2, \cdots, n-1, n+1)$. But since P_{n+1} is unique it follows that the equation

$$(17) \qquad S^c R^d_{n+1} A - R^a R^b_{n+1} B = 0$$

must be a formal identity. It must in fact be true for any numerical values of the coefficients of F_β for which these functions converge and for which R, S, R_{n+1} are different from zero, and it must be true therefore identically, since otherwise values of the coefficients could be chosen for which these conditions are satisfied and the first member of (17) is different from zero. In making this argument it is to be borne in mind that only a finite number of the coefficients of the F_β occur in A, B, R, S, R_{n+1}. From the usual algebraic elimination theory it is known that R, S, R_{n+1} are irreducible,* and evidently no two of these can be the same, since they involve coefficients of different functions. If it is presupposed that A and B are not divisible

* KÖNIG, loc. cit., p. 271.

by R_{n+1} or, respectively, by R and S, then it follows that in the identity (17) $b - d = a = c = 0$, and the only denominators in the coefficients of P_{n+1} are powers of R_{n+1}.

It has been shown that the properties (2) and (3) of §1 hold for P_{n+1} in numerical cases for which the functions F_β converge and the two resultants R and R_{n+1} are different from zero, but the restriction $R \neq 0$ is necessitated only by the method of proof, as the following considerations will show. In the first place, if $R_{n+1} \neq 0$ in any numerical case, then a linear homogeneous transformation can always be made taking $y_1, y_2, \cdots, y_{n+1}$ into $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_{n+1}$ in such a way that the new resultant $\bar{R}$ is different from zero. For since the characteristic polynomials f_β $(\beta = 1, 2, \cdots, n+1)$ have no common roots, it follows that the f_α $(\alpha = 1, 2, \cdots, n)$ can have only a finite number of roots in common,* and these can always be transformed in such a way that $\bar{y}_{n+1} \neq 0$ for all of them. Then the system $\bar{f}_\alpha(\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n, 0)$ $(\alpha = 1, 2, \cdots, n)$ can have no common root and its resultant $\bar{R}$ must be different from zero.

Suppose now that the functions F_β have literal coefficients, and let them be transformed by the substitution

$$\bar{y}_\beta = \sum_{\gamma=1}^{n+1} u_{\beta\gamma} y_\gamma, \tag{18}$$

in which the coefficients $u_{\beta\gamma}$ are new indeterminates. The functions $\bar{F}_\beta(x:\bar{y})$ $(\beta = 1, 2, \cdots, n+1)$ so found will have a polynomial

$$\bar{Q}(\bar{t}, x; z_{n+1}) = \prod_p (z_{n+1} - \bar{t}_1\bar{y}_{1p} - \bar{t}_2\bar{y}_{2p} - \cdots - \bar{t}_{n+1}\bar{y}_{n+1,\,p}),$$

with the properties described in §1, at least in numerical cases in which $\bar{R} \neq 0$. In the function

$$Q(t, x; z_{n+1}) = \prod_p (z_{n+1} - t_1 y_{1p} - t_2 y_{2p} - \cdots - t_{n+1} y_{n+1,\,p}) \tag{19}$$

found from $\bar{Q}$ by substituting

$$t_\gamma = \sum_{\beta=1}^{n+1} u_{\beta\gamma} \bar{t}_\beta, \tag{20}$$

* Otherwise there would be an infinity of roots with some one of the variables, say y_{n+1}, different from zero. At least one irreducible partial resolvent $D_t^{(h)}(y; y_{h+1}, \cdots, y_{n-1}; t_1, t_2, \cdots, t_n)$ (see König, loc. cit., p. 207) for the system $f_\alpha(y_1, y_2, \cdots, y_n, 1)$ $(\alpha = 1, 2, \cdots, n)$, for which $h < n - 1$, would be different from unity. The product

$$\prod_q f_{n+1}(y_{1q}, y_{2q}, \cdots, y_{nq}, 1),$$

where $(y_{1q}, y_{2q}, \cdots, y_{nq})$ is the system of coefficients of $t_1, t_2, \cdots, t_n$ in one of the linear factors of $D_t^{(h)}$ and the product is taken for all such factors, is symmetric in the root systems $(y_{1q}, y_{2q}, \cdots, y_{nq})$ and therefore expressible as a polynomial in $y_{h+1}, \cdots, y_{n-1}$. Any set of values $y_{h+1}, \cdots, y_{n-1}$ which makes this polynomial vanish would therefore define a root of $f_\beta(y_1, y_2, \cdots, y_n, 1) = 0$ $(\beta = 1, 2, \cdots, n+1)$, a system of equations which can have no common root if $R_{n+1} \neq 0$.

the coefficient systems $(y_{1p}, y_{2p}, \cdots, y_{np})$ will in such numerical cases be roots of the original functions F_β, and (19) must therefore be exactly the polynomial Q for those functions. It is known already that Q is convergent and determines root systems of the equations $F_\beta = 0$ in numerical cases for which $R \neq 0$. If $R = 0$ the transformation (18) can be so chosen that $\bar{R} \neq 0$, and since Q can be found from $\bar{Q}$ by applying the transformation (20), it follows that in this case also Q is convergent and has the desired properties. In the particular case when $z_{n+1} = y_{n+1}$, $t_1 = t_2 = \cdots = t_n = 0$, $t_{n+1} = 1$, the polynomial Q reduces to $P_{n+1}(x : y_{n+1})$.

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